Further Pure Mathematics 1

Complex Numbers

Section 1: Introduction to complex numbers

Solutions to Exercise

1. (i) $z^2 + 25 = 0$

$z^2 = -25$

$z = \pm 5j$

(ii) $4z^2 + 9 = 0$

$z^2 = -\frac{9}{4}$

$z = \pm\frac{3}{2}j$

(iii) $z^2 - 2z + 2 = 0$

$$z = \frac{2 \pm \sqrt{4 - 4\times1\times2}}{2}$$
$$= \frac{2 \pm \sqrt{-4}}{2}$$
$$= \frac{2 \pm 2j}{2}$$
$$= 1 \pm j$$

(iv) $4z^2 + 4z + 5 = 0$

$$z = \frac{-4 \pm \sqrt{16 - 4\times4\times5}}{8}$$
$$= \frac{-4 \pm \sqrt{-64}}{8}$$
$$= \frac{-4 \pm 8j}{8}$$
$$= -\tfrac{1}{2} \pm j$$

2. (i) (a) $z_1 + z_2 = 2 + 3j + 1 - 2j = 3 + j$

(b) $z_1 - z_2 = 2 + 3j - 1 + 2j = 1 + 5j$

(c) $z_1 z_2 = (2 + 3j)(1 - 2j) = 2 - 4j + 3j + 6 = 8 - j$

(d) $z_1{}^* = 2 - 3j$

(e) $z_2{}^* = 1 + 2j$

(f) $z_1{}^* + z_2{}^* = 2 - 3j + 1 + 2j = 3 - j$

(g) $z_1{}^* - z_2{}^* = 2 - 3j - 1 - 2j = 1 - 5j$

(h) $z_1{}^* z_2{}^* = (2-3j)(1+2j) = 2 + 4j - 3j + 6 = 8 + j$

(ii) (a) $z_1 + z_2 = -2j + 3 + j = 3 - j$

(b) $z_1 - z_2 = -2j - 3 - j = -3 - 3j$

(c) $z_1 z_2 = -2j(3+j) = -6j + 2 = 2 - 6j$

(d) $z_1{}^* = 2j$

(e) $z_2{}^* = 3 - j$

(f) $z_1{}^* + z_2{}^* = 2j + 3 - j = 3 + j$

(g) $z_1{}^* - z_2{}^* = 2j - (3 - j) = -3 + 3j$

(h) $z_1{}^* z_2{}^* = 2j(3-j) = 6j + 2 = 2 + 6j$

$z_1{}^* + z_2{}^* = (z_1 + z_2)^*$
$z_1{}^* - z_2{}^* = (z_1 - z_2)^*$
$z_1{}^* z_2{}^* = (z_1 z_2)^*$

3. $z = (a + j)^4$

$= a^4 + 4a^3 j + 6a^2 j^2 + 4a j^3 + j^4$

$= a^4 + 4a^3 j - 6a^2 - 4a j + 1$

If z is real, $4a^3 - 4a = 0$

$4a(a^2 - 1) = 0$

$4a(a+1)(a-1) = 0$

so $a = 0, -1$ or 1.

4. $(a + bj)^* = (a + bj)^2$

$a - bj = a^2 + 2abj - b^2$

Equating imaginary parts: $-b = 2ab$

$b + 2ab = 0$

$b(1 + 2a) = 0$

$b = 0$ or $a = -\frac{1}{2}$

Equating real parts: $a = a^2 - b^2$

If $b = 0$: $a = a^2$

$a(1 - a) = 0$

$a = 0$ or 1

If $a = -\frac{1}{2}$: $-\frac{1}{2} = \frac{1}{4} - b^2$

$b^2 = \frac{3}{4}$

$b = \pm\frac{1}{2}\sqrt{3}$

The possible values of a and b are: $a = b = 0$

$a = 1,\ b = 0$

$a = -\tfrac{1}{2},\ b = \pm\tfrac{1}{2}\sqrt{3}$

5. (i)
$$\frac{1}{3+2j} + \frac{1}{3-2j} = \frac{3-2j+3+2j}{(3+2j)(3-2j)}$$
$$= \frac{6}{9+4}$$
$$= \frac{6}{13}$$

(ii)
$$3 + j + \frac{4}{3-j} = 3 + j + \frac{4(3+j)}{(3-j)(3+j)}$$
$$= 3 + j + \frac{4(3+j)}{9+1}$$
$$= 3 + j + \tfrac{2}{5}(3+j)$$
$$= \tfrac{7}{5}(3+j)$$

(iii)
$$\frac{3}{1-j} - \frac{2j}{2+j} = \frac{3(1+j)}{(1-j)(1+j)} - \frac{2j(2-j)}{(2+j)(2-j)}$$
$$= \frac{3+3j}{2} - \frac{4j+2}{5}$$
$$= \frac{15+15j-8j-4}{10}$$
$$= \frac{11+7j}{10}$$

6. (i) $(a+bj)(2+j) = a - 3j$

$2a + aj + 2bj - b = a - 3j$

Equating real parts: $2a - b = a$

$a = b$

Equating imaginary parts: $a + 2b = -3$

$3a = -3$

$a = -1$

$a = -1,\ b = -1$

(ii) $(a+j)(4-bj) = 3b + 2aj$

$4a - abj + 4j + b = 3b + 2aj$

Equating real parts: $4a + b = 3b$

$2a = b$

$-ab + 4 = 2a$

$-2a^2 + 4 = 2a$

Equating imaginary parts: $a^2 + a - 2 = 0$

$(a+2)(a-1) = 0$

$a = -2$ or $a = 1$

$a = -2$, $b = -4$ or $a = 1$, $b = 2$.

7. $(a + bj)^2 = 3 - 4j$

$a^2 + 2abj - b^2 = 3 - 4j$

Equating imaginary parts: $2ab = -4$

$$b = -\frac{2}{a}$$

Equating real parts: $a^2 - b^2 = 3$

$$a^2 - \frac{4}{a^2} = 3$$

$$a^4 - 4 = 3a^2$$

$$a^4 - 3a^2 - 4 = 0$$

$$(a^2 - 4)(a^2 + 1) = 0$$

Since a is real, $a = \pm 2$ so $b = \mp 1$

The square roots of 3 – 4j are $2 - j$ and $-2 + j$.

8. (i) One root is 2 – j so the other root is 2 + j

Equation is $(z - 2 + j)(z - 2 - j) = 0$

$$(z-2)^2 + 1 = 0$$

$$z^2 - 4z + 4 + 1 = 0$$

$$z^2 - 4z + 5 = 0$$

$p = -4$, $q = 5$

(ii) One root is 1 – 3j so the other root is 1 + 3j

Equation is $(z - 1 + 3j)(z - 1 - 3j) = 0$

$$(z-1)^2 + 9 = 0$$

$$z^2 - 2z + 1 + 9 = 0$$

$$z^2 - 2z + 10 = 0$$

$p = -2$, $q = 10$

(iii) One root is 2j so the other root is -2j

Equation is $(z-2j)(z+2j)=0$

$$z^2+4=0$$

$p=0,\ q=4$

(iv) One root is $5-3j$ so the other root is $5+3j$

Equation is $(z-5+3j)(z-5-3j)=0$

$$(z-5)^2+9=0$$

$$z^2-10z+25+9=0$$

$$z^2-10z+34=0$$

$p=-10,\ q=34$

9. $$\frac{5}{a+bj}+\frac{2}{1+3j}=1$$

$$\frac{5}{a+bj}=1-\frac{2}{1+3j}=\frac{1+3j-2}{1+3j}=\frac{-1+3j}{1+3j}$$

$$\frac{a+bj}{5}=\frac{1+3j}{-1+3j}$$

$$=\frac{(1+3j)(-1-3j)}{(-1+3j)(-1-3j)}$$

$$=\frac{-1-6j+9}{1+9}$$

$$=\frac{8-6j}{10}$$

$$=\frac{4-3j}{5}$$

$$a+bj=4-3j$$